

Modified Iterated Crank-Nicolson Method with Improved Accuracy

Qiqi Tran* and Jinjie Liu†

Department of Mathematical Sciences, Delaware State University

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Abstract

The iterated Crank-Nicolson (ICN) method is a successful numerical algorithm in numerical relativity for solving partial differential equations. The θ -ICN method is the extension of the original ICN method where θ is the weight when averaging the predicted and corrected values. It has better stability when θ is chosen to be larger than 0.5, but the accuracy is reduced since the θ -ICN method is second order accurate only when $\theta = 0.5$. In this paper, we propose two modified θ -ICN algorithms that have second order of convergence rate when θ is not 0.5, based on two different ways to choose the weight θ . The first approach employs two geometrically averaged θ s in two iterations within one time step, and the second one uses arithmetically averaged θ s for two consecutive time steps while θ remains the same in each time step. The stability and second order accuracy of our methods are verified using stability and truncation error analysis and are demonstrated by numerical examples on linear and semi-linear hyperbolic partial differential equations and Burgers' equation.

* qhuang11@students.desu.edu

† jliu@desu.edu

I. INTRODUCTION

The iterated Crank-Nicolson (ICN) method is a popular and successful numerical method in numerical relativity for solving partial differential equations [1, 2]. The ICN method is the explicit version of the Crank-Nicolson (CN) method, which is a very famous implicit finite difference method for solving partial differential equations [3]. The ICN method transforms the implicit CN method into an explicit algorithm through a sequence of iterations. It has been suggested by Teukolsky [1] that one should carry out exactly two iterations and no more, since the accuracy is not affected by doing more iterations. In this paper we will consider the ICN method with two iterations.

The stability of the ICN method can be improved by introducing a variable θ , where θ is the weight when averaging the predicted and corrected values. The resulting ICN method is referred as the θ -ICN method, which was introduced by Leiler and Rezzolla in 2006 [2]. The original ICN method is the special case of θ -ICN when $\theta = 0.5$. In numerical relativity simulations, θ is usually chosen to be larger than 0.5 to obtain better stability. For instance, θ is chosen to be 0.51 in [4], and in [5, 6] the authors find that $\theta \neq 0.5$ yields an improved stability. A major drawback of using $\theta \neq 0.5$ in θ -ICN method is that the accuracy is reduced to first order. Only when $\theta = 0.5$, the θ -ICN method is second order accurate [2].

In this paper, we propose two modified θ -ICN algorithms that have second order of convergence when $\theta \neq 0.5$. The first one is based on the geometrically averaged weights of two consecutive iterations of the θ -ICN method and this method is referred as the geometric averaging (GA) θ -ICN algorithm. The second one is based on using two arithmetically averaged weights for two consecutive time steps and this method is referred as the arithmetic averaging (AA) θ -ICN algorithm. The paper is organized as follows: in section II, we review the original ICN and θ -ICN methods. In section III, we discuss the proposed modified θ -ICN algorithms with improved accuracy. Numerical examples on linear hyperbolic PDE, semi-linear hyperbolic PDE, and Burgers' equation are presented in section IV.

II. THE ICN AND θ -ICN METHODS

Consider the linear hyperbolic PDE

$$u_t + au_x = 0, \tag{1}$$

where a is a constant. The ICN method solves the implicit Crank-Nicolson update equation by iteration and turn it into an explicit scheme. First, the centered difference scheme is used to discretize the equation (1)

$$\frac{\tilde{u}_j^{n+1} - u_j^n}{\Delta t} + a \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0. \quad (2)$$

Solving for \tilde{u}_j^{n+1} , we get

$$\tilde{u}_j^{n+1} = u_j^n - a \frac{\Delta t}{2\Delta x} (u_{j+1}^n - u_{j-1}^n). \quad (3)$$

Next, an intermediate variable $\bar{u}_j^{n+1/2}$ is defined as

$$\bar{u}_j^{n+1/2} = \frac{1}{2}(\tilde{u}_j^{n+1} + u_j^n). \quad (4)$$

Apply the centered difference scheme using intermediate value $\bar{u}_j^{n+1/2}$, we get

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{\bar{u}_{j+1}^{n+1/2} - \bar{u}_{j-1}^{n+1/2}}{2\Delta x} = 0, \quad (5)$$

which can be written as

$$u_j^{n+1} = u_j^n - a \frac{\Delta t}{2\Delta x} (\bar{u}_{j+1}^{n+1/2} - \bar{u}_{j-1}^{n+1/2}). \quad (6)$$

Equations (3), (4), and (6) are the update equations of the ICN method with one iteration. For more iterations, one needs to repeat equations (3) and (4) .

For the θ -ICN method [2], we no longer weight \tilde{u}_j^{n+1} and u_j^n equally in equation (4). Instead, we define

$$\bar{u}_j^{n+1/2} = \theta \tilde{u}_j^{n+1} + (1 - \theta) u_j^n, \quad (7)$$

where $0 \leq \theta \leq 1$.

Let $R = a \frac{\Delta t}{2\Delta x}$, solving equation (1) using θ -ICN method with two iterations, we obtain the following update equations

$$^{(1)}\tilde{u}_j^{n+1} = u_j^n - R(u_{j+1}^n - u_{j-1}^n), \quad (8)$$

$$^{(1)}\bar{u}_j^{n+1/2} = \theta ^{(1)}\tilde{u}_j^{n+1} + (1 - \theta) u_j^n, \quad (9)$$

$$^{(2)}\tilde{u}_j^{n+1} = u_j^n - R(^{(1)}\bar{u}_{j+1}^{n+1/2} - ^{(1)}\bar{u}_{j-1}^{n+1/2}), \quad (10)$$

$$^{(2)}\bar{u}_j^{n+1/2} = \theta ^{(2)}\tilde{u}_j^{n+1} + (1 - \theta) u_j^n, \quad (11)$$

$$u_j^{n+1} = u_j^n - R(^{(2)}\bar{u}_{j+1}^{n+1/2} - ^{(2)}\bar{u}_{j-1}^{n+1/2}). \quad (12)$$

Leiler and Rezzolla also suggested to swap the weights θ s for θ -ICN [2]. For swapped θ -ICN method, the $\bar{u}_j^{n+1/2}$ terms in two consecutive iterations are calculated using swapped weights

$$^{(1)}\bar{u}_j^{n+1/2} = \theta^{(1)}\tilde{u}_j^{n+1} + (1 - \theta)u_j^n, \quad (13)$$

$$^{(2)}\bar{u}_j^{n+1/2} = (1 - \theta)^{(2)}\tilde{u}_j^{n+1} + \theta u_j^n. \quad (14)$$

III. MODIFIED θ -ICN METHODS

The θ -ICN method is only first order accurate when $\theta \neq 0.5$. In this section, we propose two different ways to modify θ to achieve second order accuracy when $\theta \neq 0.5$.

A. The Geometric Averaging θ -ICN method

The first idea is to take the geometric mean of two θ s in two consecutive iterations to be $\frac{1}{2}$. We name this method Geometric Averaging (GA) θ -ICN method. Taking equation (1) as an example and letting $R = a \frac{\Delta t}{2\Delta x}$, the GA θ -ICN method consists of the following five steps.

Step 1. Calculate \tilde{u}_j^{n+1}

$$\tilde{u}_j^{n+1} = u_j^n - R(u_{j+1}^n - u_{j-1}^n). \quad (15)$$

Step 2. Average \tilde{u}_j^{n+1} and u_j^n using weight θ_1 to obtain $\bar{u}_j^{n+\theta_1}$

$$\bar{u}_j^{n+\theta_1} = \theta_1 \tilde{u}_j^{n+1} + (1 - \theta_1)u_j^n. \quad (16)$$

Step 3. Calculate $\tilde{u}_j^{n+2\theta_1}$ using $\bar{u}_j^{n+\theta_1}$

$$\tilde{u}_j^{n+2\theta_1} = u_j^n - 2\theta_1 R(\bar{u}_{j+1}^{n+\theta_1} - \bar{u}_{j-1}^{n+\theta_1}). \quad (17)$$

Step 4. Average again using weight θ_2 to obtain $\bar{u}_j^{n+1/2}$

$$\bar{u}_j^{n+1/2} = \theta_2 \tilde{u}_j^{n+2\theta_1} + (1 - \theta_2)u_j^n. \quad (18)$$

Step 5. Compute u_j^{n+1} using $\bar{u}_j^{n+1/2}$

$$u_j^{n+1} = u_j^n - R(\bar{u}_{j+1}^{n+1/2} - \bar{u}_{j-1}^{n+1/2}). \quad (19)$$

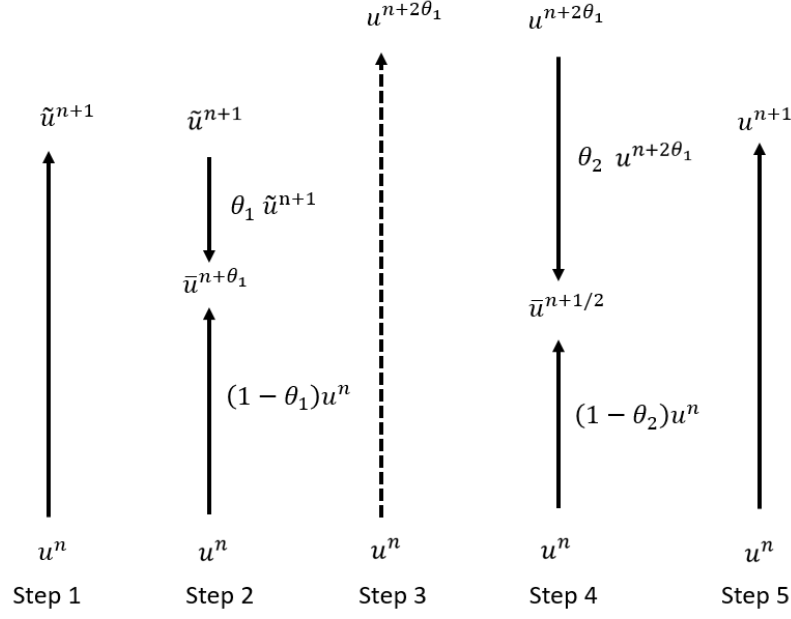


FIG. 1. Five steps for the GA θ -ICN method.

We define θ_1 and θ_2 to be positive real numbers and their geometric mean to be $\frac{1}{2}$, that is $\sqrt{\theta_1\theta_2} = \frac{1}{2}$, or equivalently $\theta_1\theta_2 = \frac{1}{4}$.

Figure 1 shows the steps of the GA θ -ICN method. In step 2, the averaged value $\bar{u}_j^{n+\theta_1}$ is located at time level $n + \theta_1$ (no longer at $n + \frac{1}{2}$ if $\theta_1 \neq 0.5$). In step 3, the solution is updated from n to $n + 2\theta_1$, in order to maintain the centered difference in time. Note that equation (17) is obtained from

$$\frac{\tilde{u}_j^{n+2\theta_1} - u_j^n}{2\theta_1\Delta t} + a \frac{\bar{u}_{j+1}^{n+\theta_1} - u_{j-1}^{n+\theta_1}}{2\Delta x} = 0, \quad (20)$$

where both u_t and u_x are approximated using centered difference. In step 4, in order to compute the averaged solution at $n + \frac{1}{2}$ as in equation (18), we have

$$n + \frac{1}{2} = \theta_2(n + 2\theta_1) + (1 - \theta_2)n = n + 2\theta_1\theta_2. \quad (21)$$

Solving equation (21), we get $\theta_1\theta_2 = 1/4$. Note that when $\theta_1 = 0.5$ and $\theta_2 = 1/(4\theta_1) = 0.5$, the GA θ -ICN method becomes the standard ICN method (θ -ICN with $\theta = 0.5$).

Combining steps 1 to 5, we have

$$\begin{aligned}
u_j^{n+1} = & -2\theta_1^2\theta_2R^3u_{j+3}^n + 2\theta_1\theta_2R^2u_{j+2}^n + (6\theta_1^2\theta_2R^3 - R) \\
& u_{j+1}^n + (1 - 4\theta_1\theta_2R^2)u_j^n + (R - 6\theta_1^2\theta_2R^3)u_{j-1}^n \\
& + 2\theta_1\theta_2R^2u_{j-2}^n + 2\theta_1^2\theta_2R^3u_{j-3}^n.
\end{aligned} \tag{22}$$

Apply the von Neumann stability analysis [1], and let

$$u_j^n = \xi^n e^{ikj\Delta x}, \tag{23}$$

we get the amplification factor

$$g(\xi) = 1 - 2\beta i - 2\beta^2 + 4\theta_1\beta^3 i, \tag{24}$$

where $\beta = R \sin(k\Delta x)$.

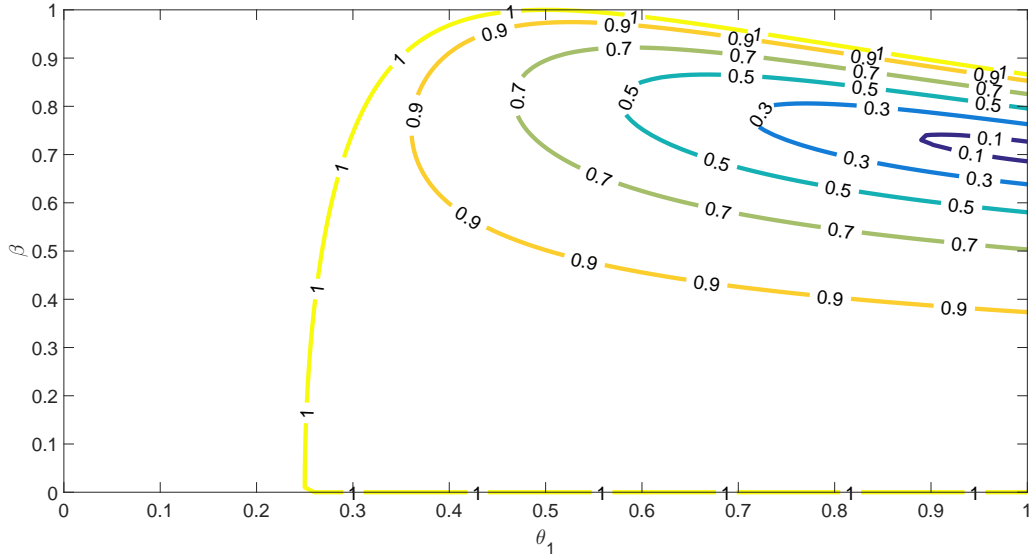


FIG. 2. (Color online) Stability Region for the GA θ -ICN method.

Figure 2 shows the stability region in the (θ_1, β) plane for the GA θ -ICN method. We see that θ_1 is no longer required to be greater than 0.5 for the method to be stable.

In the following, we calculate the truncation error of the GA θ -ICN method. Let

$$\delta^1 u_j^n = u_{j+1}^n - u_{j-1}^n, \tag{25}$$

$$\delta^2 u_j^n = u_{j+2}^n - 2u_j^n + u_{j-2}^n, \tag{26}$$

$$\delta^3 u_j^n = u_{j+3}^n - 3u_{j+1}^n + 3u_{j-1}^n - u_{j-3}^n. \tag{27}$$

Equation (22) can be rewritten as:

$$u_j^{n+1} = u_j^n - R\delta^1 u_j^n + 2\theta_1\theta_2 R^2 \delta^2 u_j^n - 2\theta_1^2\theta_2 R^3 \delta^3 u_j^n. \quad (28)$$

Use the Taylor expansions, $\delta^1 u_j^n$, $\delta^2 u_j^n$, and $\delta^3 u_j^n$ become

$$\delta^1 u_j^n = 2\Delta x (u_x)_j^n + O(\Delta x^3), \quad (29)$$

$$\delta^2 u_j^n = 4\Delta x^2 (u_{xx})_j^n + O(\Delta x^4), \quad (30)$$

$$\delta^3 u_j^n = O(\Delta x^3). \quad (31)$$

Substituting equations (29), (30) and (31) into equation (28), and apply $R = a\frac{\Delta t}{2\Delta x}$, we obtain

$$u_j^{n+1} = u_j^n - a\Delta t (u_x)_j^n + 2\theta_1\theta_2 a^2 \Delta t^2 (u_{xx})_j^n + O(\Delta x^3). \quad (32)$$

The Taylor expansion of u_j^{n+1} gives

$$u_j^{n+1} = u_j^n + \Delta t (u_t)_j^n + \frac{\Delta t^2}{2} (u_{tt})_j^n + O(\Delta t^3). \quad (33)$$

The resulting truncation error is

$$e_\tau = \frac{\Delta t}{2} u_{tt} - 2\theta_1\theta_2 a^2 \Delta t u_{xx} + O(\Delta t^2) + O(\Delta x^2). \quad (34)$$

The original differential equation (1) implies that $u_{tt} = a^2 u_{xx}$, so

$$e_\tau = 2\left(\frac{1}{4} - \theta_1\theta_2\right) a^2 u_{xx} \Delta t + O(\Delta t^2) + O(\Delta x^2). \quad (35)$$

Since $\theta_1\theta_2 = \frac{1}{4}$ for the GA θ -ICN method, the first term on the right hand side of the equation (35) vanishes and the truncation error becomes $e_\tau = O(\Delta t^2) + O(\Delta x^2)$. This proves that the GA θ -ICN method is second order accurate in both time and space.

B. The Arithmetic Averaging θ -ICN method

The second idea is to modify θ in different time steps. We define the θ in odd time steps to be θ_o , the θ in even time steps to be θ_e , and we require $\theta_o + \theta_e = 1$. We call this method the Arithmetic Averaging (AA) θ -ICN method, because the arithmetic mean of θ_o and θ_e is $\frac{1}{2}$.

Consider the linear hyperbolic equation (1) and let $R = a \frac{\Delta t}{2\Delta x}$. For the odd time step, we have

$$^{(1)}\tilde{u}_j^{n+1} = u_j^n - R(u_{j+1}^n - u_{j-1}^n), \quad (36)$$

$$^{(1)}\bar{u}_j^{n+1/2} = \theta_o^{(1)}\tilde{u}_j^{n+1} + (1 - \theta_o)u_j^n, \quad (37)$$

$$^{(2)}\tilde{u}_j^{n+1} = u_j^n - R(^{(1)}\bar{u}_{j+1}^{n+1/2} - ^{(1)}\bar{u}_{j-1}^{n+1/2}), \quad (38)$$

$$^{(2)}\bar{u}_j^{n+1/2} = \theta_o^{(2)}\tilde{u}_j^{n+1} + (1 - \theta_o)u_j^n, \quad (39)$$

$$u_j^{n+1} = u_j^n - R(^{(2)}\bar{u}_{j+1}^{n+1/2} - ^{(2)}\bar{u}_{j-1}^{n+1/2}). \quad (40)$$

For the next time step (which is an even time step)

$$^{(1)}\tilde{u}_j^{n+2} = u_j^{n+1} - R(u_{j+1}^{n+1} - u_{j-1}^{n+1}), \quad (41)$$

$$^{(1)}\bar{u}_j^{n+3/2} = \theta_e^{(1)}\tilde{u}_j^{n+2} + (1 - \theta_e)u_j^{n+1}, \quad (42)$$

$$^{(2)}\tilde{u}_j^{n+2} = u_j^{n+1} - R(^{(1)}\bar{u}_{j+1}^{n+3/2} - ^{(1)}\bar{u}_{j-1}^{n+3/2}), \quad (43)$$

$$^{(2)}\bar{u}_j^{n+3/2} = \theta_e^{(2)}\tilde{u}_j^{n+2} + (1 - \theta_e)u_j^{n+1}, \quad (44)$$

$$u_j^{n+2} = u_j^{n+1} - R(^{(2)}\bar{u}_{j+1}^{n+3/2} - ^{(2)}\bar{u}_{j-1}^{n+3/2}). \quad (45)$$

Note that when $\theta_o = \theta_e = 0.5$, the AA θ -ICN method becomes the standard ICN method (θ -ICN with $\theta = 0.5$).

Substituting equations (36) - (39) into equation (40), we obtain

$$\begin{aligned} u_j^{n+1} = & -\theta_o^2 R^3 u_{j+3}^n + \theta_o R^2 u_{j+2}^n + (3\theta_o^2 R^3 - R)u_{j+1}^n \\ & + (1 - 2\theta_o R^2)u_j^n + (R - 3\theta_o^2 R^3)u_{j-1}^n \\ & + \theta_o R^2 u_{j-2}^n + \theta_o^2 R^3 u_{j-3}^n. \end{aligned}$$

We can find the amplification factor $g_o(\xi)$ for the odd time step

$$g_o(\xi) = 1 - 2\beta i - 4\theta_o \beta^2 + 8\theta_o^2 \beta^3 i. \quad (46)$$

Similarly, the amplification factor $g_e(\xi)$ for the even time step is

$$g_e(\xi) = 1 - 2\beta i - 4\theta_e \beta^2 + 8\theta_e^2 \beta^3 i. \quad (47)$$

The product of two amplification factors $g_o(\xi)$ and $g_e(\xi)$ is the amplification factor from time steps n to $n + 2$. Figure 3 shows the stability region in (θ_o, β) plane. We see that

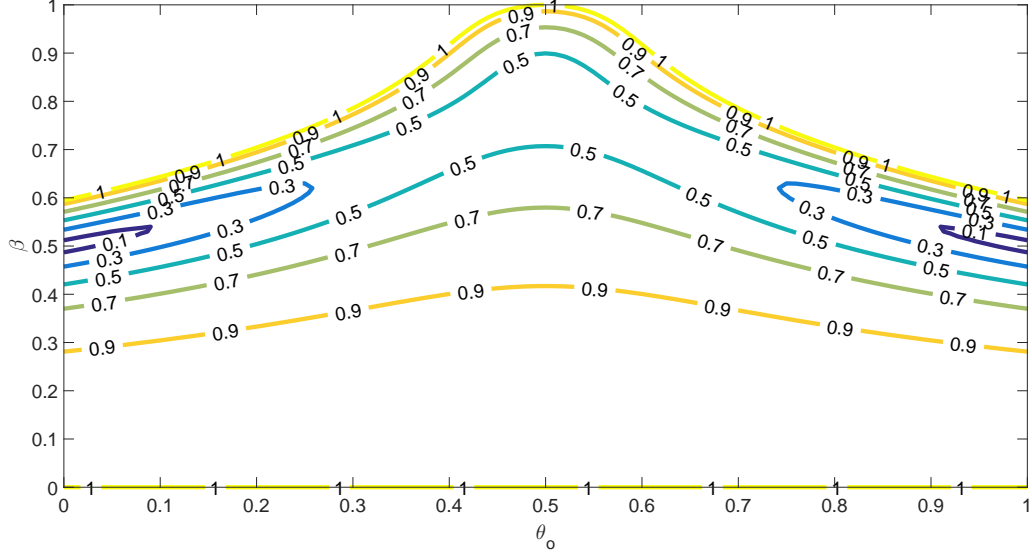


FIG. 3. (Color online) Stability Region for the AA θ -ICN method.

the stability region is symmetric with respect to $\theta_o = 0.5$. In comparison to the GA θ -ICN method, the AA θ -ICN method has slightly larger stability region, but slightly more damping. For example, when $\theta_o = 0.4$ and $\beta = 0.6$, the AA θ -ICN method has an amplification factor between 0.5 and 0.7, but the GA θ -ICN method has a larger amplification factor around 0.9.

Following a similar procedure as the GA θ -ICN method, we find that

$$u_j^{n+1} = u_j^n - a\Delta t(u_x)_j^n + a^2\theta_o\Delta t^2(u_{xx})_j^n + O(\Delta x^3), \quad (48)$$

and

$$u_j^{n+2} = u_j^{n+1} - a\Delta t(u_x)_j^{n+1} + a^2\theta_e\Delta t^2(u_{xx})_j^{n+1} + O(\Delta x^3). \quad (49)$$

Substitute equation (48) into equation (49), we obtain

$$\begin{aligned} u_j^{n+2} = & u_j^n - a\Delta t(u_x)_j^n + a^2\theta_o\Delta t^2(u_{xx})_j^n - a\Delta t(u_x)_j^{n+1} \\ & + a^2\theta_e\Delta t^2(u_{xx})_j^{n+1} + O(\Delta x^3). \end{aligned} \quad (50)$$

The Taylor expansions of $(u_x)_j^{n+1}$ and $(u_{xx})_j^{n+1}$ give

$$(u_x)_j^{n+1} = (u_x)_j^n + \Delta t(u_{xt})_j^n + O(\Delta t^2), \quad (51)$$

$$(u_{xx})_j^{n+1} = (u_{xx})_j^n + O(\Delta t). \quad (52)$$

Substitute equations (51) and (52) into equation (50), we get

$$\begin{aligned} u_j^{n+2} = & u_j^n - 2a\Delta t(u_x)_j^n + a^2\theta_o\Delta t^2(u_{xx})_j^n - a\Delta t^2(u_{xt})_j^n \\ & + a^2\theta_e\Delta t^2(u_{xx})_j^n + O(\Delta t^3) + O(\Delta x^3). \end{aligned} \quad (53)$$

The original differential equation (1) implies that $u_{xt} = -au_{xx}$, so we have

$$\begin{aligned} u_j^{n+2} = & u_j^n - 2a\Delta t(u_x)_j^n + a^2\Delta t^2(1 + \theta_o + \theta_e)(u_{xx})_j^n \\ & + O(\Delta x^3) + O(\Delta t^3). \end{aligned} \quad (54)$$

The Taylor expansion of u_j^{n+2} gives

$$u_j^{n+2} = u_j^n + 2\Delta t u_t + \frac{(2\Delta t)^2}{2} u_{tt} + O(\Delta t^3). \quad (55)$$

Substituting equation (55) into equation (54), we get

$$u_t + au_x = -\Delta t u_{tt} + \frac{a^2\Delta t}{2}(1 + \theta_o + \theta_e)u_{xx} + O(\Delta t^2) + O(\Delta x^2). \quad (56)$$

The original differential equation (1) implies that $u_{tt} = a^2u_{xx}$, so we obtain the truncation error

$$e_\tau = \left(\frac{1 + \theta_o + \theta_e}{2} - 1\right)a^2u_{xx}\Delta t + O(\Delta t^2) + O(\Delta x^2). \quad (57)$$

Since the AA θ -ICN method requires $\theta_o + \theta_e = 1$, the first term on the right hand side of the equation (57) vanishes. Therefore, the truncation error becomes $e_\tau = O(\Delta t^2) + O(\Delta x^2)$, which indicates that the AA θ -ICN method with two iterations is second order accurate in both time and space.

IV. NUMERICAL EXAMPLES

A. Linear hyperbolic PDE

In the first numerical example, we consider the following linear hyperbolic equation initial value problem with periodic boundary condition

$$u_t + u_x = 0, x \in [0, 1], t \in [0, 1], \quad (58)$$

$$u(0, x) = \sin^2(\pi x), u(x + 1, t) = u(x, t). \quad (59)$$

The exact solution is $u(x, t) = \sin^2((x - t)\pi)$. The Courant-Friedrichs-Lewy (CFL) condition is chosen to be 0.5 and the numbers of grid points in space are chosen to be $N = 100, 200, 400, 800$, and 1600. For the GA θ -ICN method, we chose $\theta_1 = 0.6$ and $\theta_2 = \frac{1}{4\theta_1} \approx 0.416$. For the AA θ -ICN method, we use $\theta_o = 0.6$ and $\theta_e = 0.4$. We compare our new methods with the standard ICN method, the θ -ICN method with $\theta = 0.6$, and the swapped θ -ICN method.

Figure 4 shows the numerical results and the exact solution. We see that the GA θ -ICN method, the AA θ -ICN method, and the ICN method are more accurate than the swapped θ -ICN method and the θ -ICN method with $\theta = 0.6$. We calculate the L_1 , L_2 , and L_∞ norms of each method and the results are shown in tables I, II, and III, respectively. From these tables, we see that the swapped θ -ICN method and the θ -ICN method with $\theta = 0.6$ are only first order accurate, while the ICN method, the GA and AA θ -ICN methods are second order accurate.

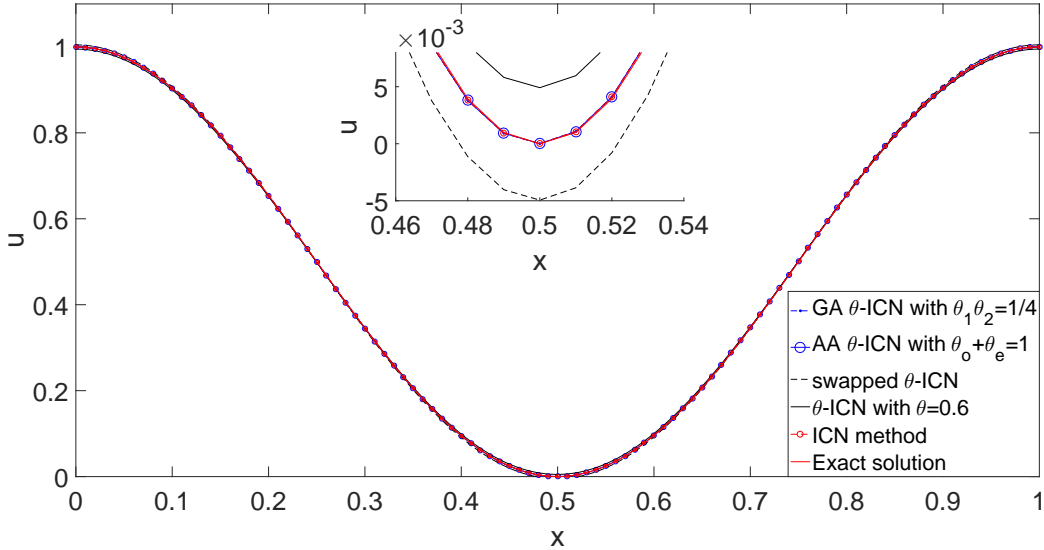


FIG. 4. (Color online) Comparison between the GA θ -ICN method, the AA θ -ICN method, the swapped θ -ICN method, the θ -ICN method ($\theta = 0.6$), the ICN method, and the exact solution for linear hyperbolic PDE.

TABLE I. Comparison of L_1 Norm for Linear Hyperbolic PDE.

	ICN		θ -ICN		Swapped θ -ICN		GA θ -ICN		AA θ -ICN	
	$\theta = 0.5$		$\theta = 0.6$		$\theta_1 + \theta_2 = 1$		$\theta_1\theta_2 = 1/4$		$\theta_o + \theta_e = 1$	
N	L_1	order	L_1	order	L_1	order	L_1	order	L_1	order
200	1.8E-4		1.6E-3		1.6E-3		2.0E-4		1.9E-4	
400	4.6E-5	2.0	7.9E-4	1.0	7.9E-4	1.0	4.9E-5	2.0	4.7E-5	2.0
800	1.2E-5	2.0	3.9E-4	1.0	3.9E-4	1.0	1.2E-5	2.0	1.2E-5	2.0
1600	2.9E-6	2.0	2.0E-4	1.0	2.0E-4	1.0	3.1E-6	2.0	2.9E-6	2.0

TABLE II. Comparison of L_2 Norm for Linear Hyperbolic PDE.

	ICN		θ -ICN		Swapped θ -ICN		GA θ -ICN		AA θ -ICN	
	$\theta = 0.5$		$\theta = 0.6$		$\theta_1 + \theta_2 = 1$		$\theta_1\theta_2 = 1/4$		$\theta_o + \theta_e = 1$	
N	L_2	order	L_2	order	L_2	order	L_2	order	L_2	order
200	1.5E-5		1.3E-4		1.3E-4		1.5E-5		1.5E-5	
400	2.6E-6	2.5	4.4E-5	1.5	4.4E-5	1.5	2.7E-6	2.5	2.6E-6	2.5
800	4.5E-7	2.5	1.5E-5	1.5	1.6E-5	1.5	4.8E-7	2.5	4.6E-7	2.5
1600	8.0E-8	2.5	5.5E-6	1.5	5.5E-6	1.5	8.6E-8	2.5	8.1E-8	2.5

B. Semi-linear hyperbolic PDE

In the second example, we consider a semi-linear hyperbolic PDE

$$u_t + u_x = -u^2, x \in [0, 1], t \in [0, 1], \quad (60)$$

$$u(0, x) = \sin^2(\pi x), u(x + 1, t) = u(x, t). \quad (61)$$

The exact solution is given by [7]

$$u(x, t) = \frac{\sin^2((x - t)\pi)}{1 + t \sin^2((x - t)\pi)}. \quad (62)$$

We use the same grid and the CFL condition as in the previous example. We solve this PDE by the GA θ -ICN method with $\theta_1 = 0.6$, the AA θ -ICN method with $\theta_o = 0.6$, the θ -ICN method with $\theta = 0.6$, the swapped θ -ICN method, and the ICN method.

TABLE III. Comparison of L_∞ Norm for Linear Hyperbolic PDE.

	ICN		θ -ICN		Swapped θ -ICN		GA θ -ICN		AA θ -ICN	
	$\theta = 0.5$		$\theta = 0.6$		$\theta_1 + \theta_2 = 1$		$\theta_1\theta_2 = 1/4$		$\theta_o + \theta_e = 1$	
N	L_∞	order	L_∞	order	L_∞	order	L_∞	order	L_∞	order
200	2.9E-4		2.5E-3		2.5E-3		3.1E-4		3.0E-4	
400	7.3E-5	2.0	1.2E-3	1.1	1.2E-3	1.1	7.8E-5	2.0	7.4E-5	2.0
800	1.8E-5	2.0	6.2E-4	1.0	6.2E-4	1.0	2.0E-5	2.0	1.8E-5	2.0
1600	4.5E-6	2.0	3.1E-4	1.0	3.2E-4	1.0	4.8E-6	2.0	4.6E-6	2.0

Figure 5 shows the results and we see that all methods give correct solutions. From the enlarged figure, we can see that the GA and AA θ -ICN methods have similar accuracy as the ICN method, while they are more accurate than the θ -ICN method with $\theta = 0.6$ and swapped θ -ICN method. We also calculate the L_1 , L_2 , and L_∞ norms to test the convergence of our proposed methods. Tables IV, V, and VI show the L_1 , L_2 , and L_∞ norm, respectively. From these tables, we see that the GA and AA θ -ICN methods are second order accurate. In comparison, the swapped θ -ICN method and the θ -ICN method with $\theta = 0.6$ are only first order accurate.

TABLE IV. Comparison of L_1 Norm for Semi-Linear Hyperbolic PDE.

	ICN		θ -ICN		Swapped θ -ICN		GA θ -ICN		AA θ -ICN	
	$\theta = 0.5$		$\theta = 0.6$		$\theta_1 + \theta_2 = 1$		$\theta_1\theta_2 = 1/4$		$\theta_o + \theta_e = 1$	
N	L_1	order	L_1	order	L_1	order	L_1	order	L_1	order
200	1.3E-4		1.1E-3		1.1E-3		1.4E-4		1.3E-4	
400	3.3E-5	2.0	5.4E-4	1.0	5.3E-4	1.1	3.5E-5	2.0	3.3E-5	2.0
800	8.1E-6	2.0	2.7E-4	1.0	2.7E-4	1.0	8.7E-6	2.0	8.2E-6	2.0
1600	2.0E-6	2.0	1.4E-4	1.0	1.3E-4	1.0	2.2E-6	2.0	2.1E-6	2.0

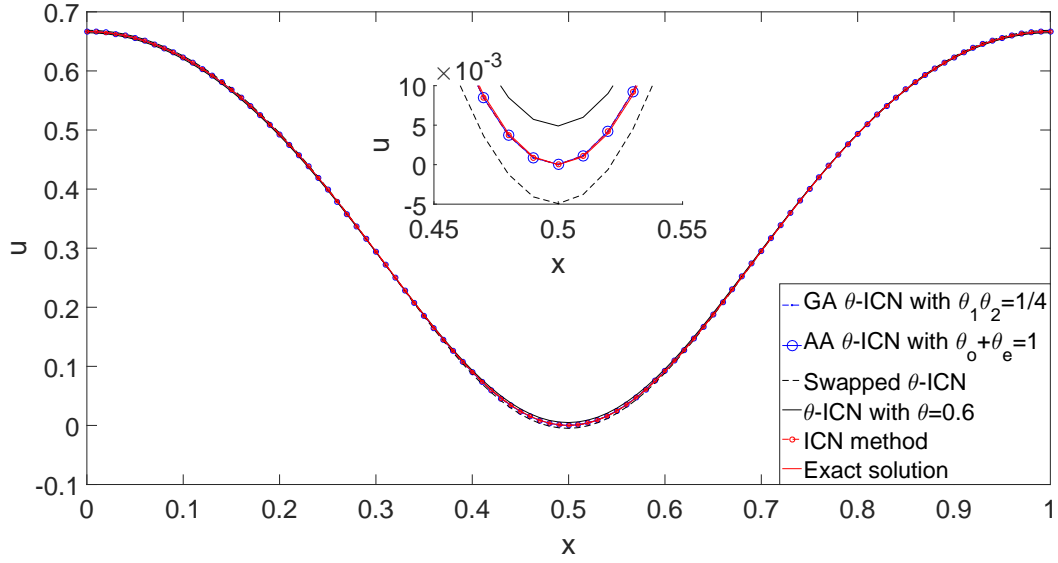


FIG. 5. (Color online) Comparison between the GA θ -ICN method, the AA θ -ICN method, the θ -ICN method ($\theta = 0.6$), the swapped θ -ICN method, the ICN method, and the exact solution for Semi-linear Hyperbolic PDE.

TABLE V. Comparison of L_2 Norm for Semi-Linear Hyperbolic PDE.

	ICN		θ -ICN		Swapped θ -ICN		GA θ -ICN		AA θ -ICN	
	$\theta = 0.5$		$\theta = 0.6$		$\theta_1 + \theta_2 = 1$		$\theta_1 \theta_2 = 1/4$		$\theta_o + \theta_e = 1$	
N	L_2	order	L_2	order	L_2	order	L_2	order	L_2	order
200	1.1E-5		9.2E-5		8.9E-5		1.2E-5		1.1E-5	
400	2.0E-6	2.5	3.2E-5	1.5	3.2E-5	1.5	2.1E-6	2.5	2.0E-6	2.5
800	3.5E-7	2.5	1.1E-5	1.5	1.1E-5	1.5	3.7E-7	2.5	3.5E-7	2.5
1600	6.1E-8	2.5	4.0E-6	1.5	4.0E-6	1.5	6.5E-8	2.5	6.2E-8	2.5

C. Burgers' equation

In the third example, we consider the Burgers' equation

$$u_t + uu_x = au_{xx}, x \in [0, 1], t \in [0, 1], \quad (63)$$

$$u(x, 0) = \sin^2(\pi x), u(x + 1, t) = u(x, t), \quad (64)$$

TABLE VI. Comparison of L_∞ Norm for Semi-Linear Hyperbolic PDE.

	ICN		θ -ICN		Swapped θ -ICN		GA θ -ICN		AA θ -ICN	
	$\theta = 0.5$		$\theta = 0.6$		$\theta_1 + \theta_2 = 1$		$\theta_1 \theta_2 = 1/4$		$\theta_o + \theta_e = 1$	
N	L_∞	order	L_∞	order	L_∞	order	L_∞	order	L_∞	order
200	2.7E-4		2.5E-3		2.5E-3		2.9E-4		2.7E-4	
400	6.7E-5	2.0	1.2E-3	1.1	1.2E-3	1.1	7.2E-5	2.0	6.8E-5	2.0
800	1.7E-5	2.0	6.2E-4	1.0	6.2E-4	1.0	1.8E-5	2.0	1.7E-5	2.0
1600	4.2E-6	2.0	3.1E-4	1.0	3.2E-4	1.0	4.5E-6	2.0	4.3E-6	2.0

where a is chosen to be 0.01 in our simulation. The term uu_x can be written in conservation form $(F(u))_x$, where $F(u) = \frac{1}{2}u^2$. The term u_{xx} can be approximated by the centered difference

$$u_{xx} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}. \quad (65)$$

We let the grid size to be $N = 30$. To test the convergence rate in temporal domain, we refine Δt and keep Δx the same in our simulations, so we let Δt to be Δ , $\Delta/2$, $\Delta/4$, and $\Delta/8$, where $\Delta = 0.5\Delta x^2$. We use the ICN method with $\Delta t = \Delta/32$ as the exact solution. Similar to previous examples, we compare the numerical results of the GA θ -ICN method with $\theta_1 = 0.6$, the AA θ -ICN method with $\theta_o = 0.6$, the θ -ICN method with $\theta = 0.6$, the swapped θ -ICN method, and the ICN method. Figure 6 shows the results and we see that the GA and AA θ -ICN methods are very close to the ICN method and they are more accurate than the θ -ICN with $\theta = 0.6$ and the swapped θ -ICN method. Tables VII, VIII and IX show the numerical results on the L_1 , L_2 and L_∞ norms, respectively. We see that the GA and the AA θ -ICN methods are second order accurate in time, while the swapped θ -ICN method and the θ -ICN method with $\theta=0.6$ are only first order accurate in time. From this example, we see that the GA and AA θ -ICN methods are suitable for solving nonlinear mixed hyperbolic-parabolic equations with improved accuracy.

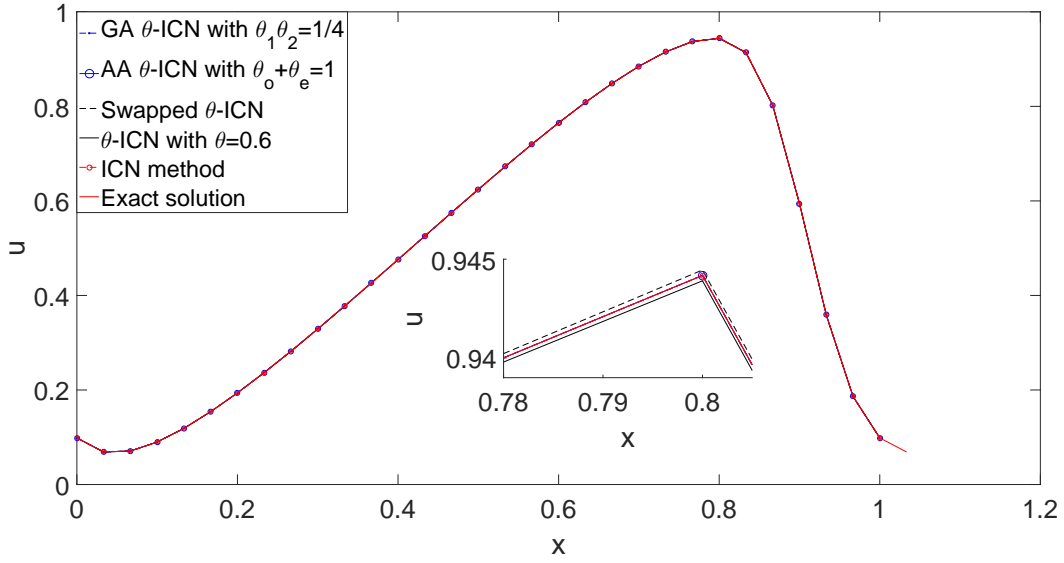


FIG. 6. (Color online) Comparison between the GA θ -ICN method, the AA θ -ICN method, the swapped θ -ICN method, the θ -ICN method ($\theta = 0.6$), the ICN method, and the exact solution for Burgers' Equation.

TABLE VII. Comparison of L_1 Norm in time for Burgers' equation.

	ICN		θ -ICN		Swapped θ -ICN		GA θ -ICN		AA θ -ICN	
	$\theta = 0.5$		$\theta = 0.6$		$\theta_1 + \theta_2 = 1$		$\theta_1\theta_2 = 1/4$		$\theta_o + \theta_e = 1$	
Δt	L_1	order	L_1	order	L_1	order	L_1	order	L_1	order
Δ	2.9E-7		7.8E-5		7.8E-5		4.7E-7		3.4E-7	
$\Delta/2$	7.3E-8	2.0	3.9E-5	1.0	3.9E-5	1.0	1.2E-7	2.0	8.5E-8	2.0
$\Delta/4$	1.8E-8	2.0	1.9E-5	1.0	1.9E-5	1.0	2.9E-8	2.0	2.1E-8	2.0
$\Delta/8$	4.3E-9	2.1	9.7E-6	1.0	9.7E-6	1.0	7.1E-9	2.0	5.0E-9	2.1

V. CONCLUSION

In this paper, we have proposed two approaches to improve the θ -iterated Crank-Nicolson (ICN) method to second order accuracy when θ does not equal to 0.5. The first approach employs geometrically averaged θ s in two iterations within one time step. The second approach uses arithmetically averaged θ s for two consecutive time steps while same θ is used

TABLE VIII. Comparison of L_2 Norm in time for Burgers' equation.

	ICN		θ -ICN		Swapped θ -ICN		GA θ -ICN		AA θ -ICN	
	$\theta = 0.5$		$\theta = 0.6$		$\theta_1 + \theta_2 = 1$		$\theta_1\theta_2 = 1/4$		$\theta_o + \theta_e = 1$	
Δt	L_2	order	L_2	order	L_2	order	L_2	order	L_2	order
Δ	9.0E-8		2.0E-5		2.0E-5		1.4E-7		1.0E-7	
$\Delta/2$	2.3E-8	2.0	1.0E-5	1.0	1.0E-5	1.0	3.6E-8	2.0	2.6E-8	1.9
$\Delta/4$	5.6E-9	2.0	5.0E-6	1.0	5.0E-6	1.0	8.9E-9	2.0	6.3E-9	2.0
$\Delta/8$	1.3E-9	2.1	2.5E-6	1.0	2.5E-6	1.0	2.2E-9	2.0	1.5E-9	2.1

TABLE IX. Comparison of L_∞ Norm in time for Burgers' equation.

	ICN		θ -ICN		Swapped θ -ICN		GA θ -ICN		AA θ -ICN	
	$\theta = 0.5$		$\theta = 0.6$		$\theta_1 + \theta_2 = 1$		$\theta_1\theta_2 = 1/4$		$\theta_o + \theta_e = 1$	
Δt	L_∞	order	L_∞	order	L_∞	order	L_∞	order	L_∞	order
Δ	1.7E-6		3.3E-4		3.4E-4		2.7E-6		1.8E-6	
$\Delta/2$	4.2E-7	2.0	1.7E-4	1.0	1.7E-4	1.0	6.7E-7	2.0	4.6E-7	2.0
$\Delta/4$	1.0E-7	2.1	8.4E-5	1.0	8.4E-5	1.0	1.7E-7	2.0	1.1E-7	2.1
$\Delta/8$	2.5E-8	2.0	4.2E-5	1.0	4.2E-5	1.0	4.0E-8	2.1	2.7E-8	2.0

in every iteration of each time step. Stability and truncation error analysis have been carried out to show that our methods are stable and second order accurate. Numerical examples on linear hyperbolic PDE, semi-linear hyperbolic PDE, and Burgers' equation are presented to verify that the second order accuracy of the proposed new methods.

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